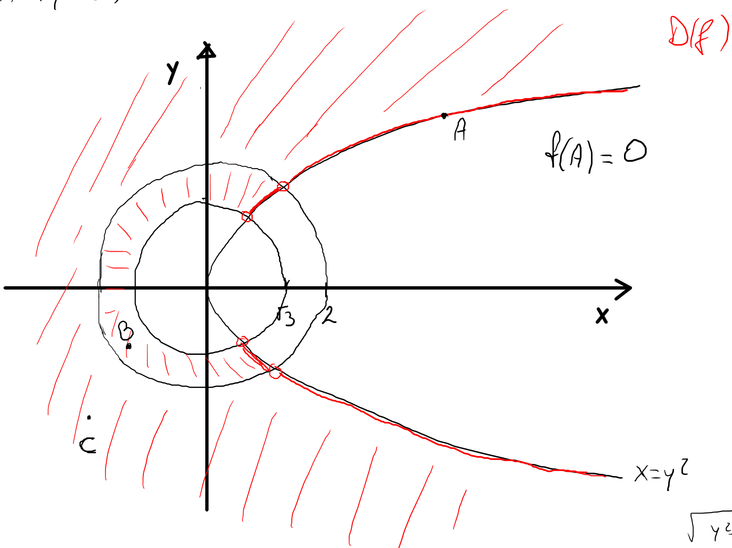
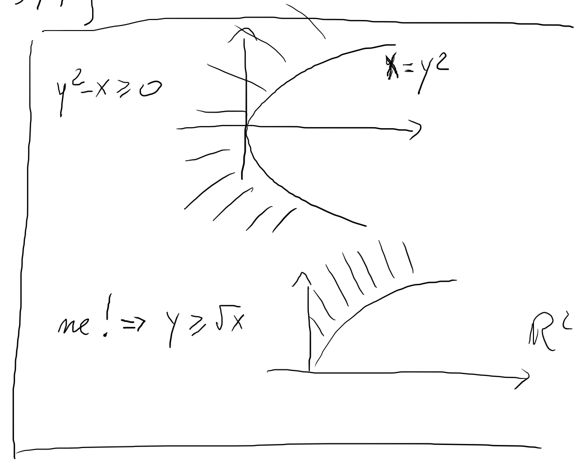
Dú 1.

$$f(x,y)=\frac{\sqrt{y^2-x}}{\ln(x^2+y^2-3)}$$

$D(f)=\{(x,y)\in\mathbb{R}^2:\ y^2-x\geq 0,\ x^2+y^2-3>0,\ x^2+y^2-3\neq 1\}$

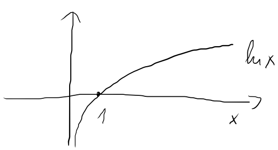




pro B $: 0<x^2+y^2-3<1,\ -\infty<\ln(x^2+y^2-3)<0 \quad 0>\dfrac{\sqrt{y^2-x}}{\ln(x^2+y^2-3)}>-\infty$

pro C $: x^2+y^2-3>1,\ 0<\ln(x^2+y^2-3)<+\infty,\ \infty>\dfrac{\sqrt{y^2-x}}{\ln(x^2+y^2-3)}>0$

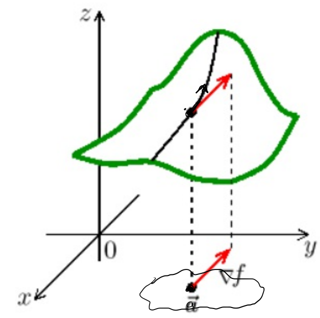
$f(A)=0 \Rightarrow H(f)=\mathbb{R}.$



Najděte jednotkový směr největšího růstu funkce f v bodě a:

(a) $f(x,y) = x^2y + e^{xy}\sin y,\ \ a = (1,0)$,

(b) $f(x,y,z) = xe^y + z^2,\ \ a = (1, \ln 2, \frac{1}{2})$.



gradient $\nabla f(\vec{a})$ udává směr největšího růstu funkce v bodě $\vec{a}$, funkce tam roste rychlostí $\|\nabla f(\vec{a})\|$.

$\operatorname{grad} f(\vec{a}) = \nabla f(\vec{a})$

a) $\nabla f|_{\vec{a}}$ $\qquad \vec{u} = \dfrac{\nabla f|_{\vec{a}}}{\|\nabla f|_{\vec{a}}\|}$

$$\frac{\partial f}{\partial \vec{u}}\Big|_{\vec{a}} = \operatorname{grad} f|_{\vec{a}} \cdot \vec{u} = \|\nabla f|_{\vec{a}}\|$$

$$\nabla f = \langle 2xy + ye^{xy}\sin y,\ x^2 + xe^{xy}\sin y + e^{xy}\cos y \rangle$$

$\nabla f|_{(1,0)} = \langle 0, 2 \rangle \qquad \vec{u} = \langle 0, 1 \rangle$

$$\frac{\partial f}{\partial \vec{u}}\Big|_{\vec{a}} = 2$$

b) $\nabla f = \langle e^y, xe^y, 2z \rangle$.

$\nabla f|_{\vec{a}} = \langle 2, 2, 1 \rangle \qquad \vec{u} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$

$$\|\nabla f|_{\vec{a}}\| = \sqrt{4+4+1} = 3$$

Nalezněte první diferenciál d $f(a)$, gradient grad $f(a)$, rovnice tečné roviny a normály ke grafu funkce $f(x,y) = \sqrt{x - \sqrt{y}}$ v bodech $a = (2,1)$ a $b = (0,9)$. ?

$b \notin D(f)$

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$f(2,1) = 1$

$$\operatorname{grad} f = \left\langle \frac{1}{2\sqrt{x-\sqrt{y}}}, \frac{-\frac{1}{2\sqrt{y}}}{2\sqrt{x-\sqrt{y}}} \right\rangle = \left\langle \frac{1}{2\sqrt{x-\sqrt{y}}}, \frac{-1}{4\sqrt{y}\sqrt{x-\sqrt{y}}} \right\rangle$$

$$\operatorname{grad} f|_a = \left\langle \frac{1}{2}, \frac{-1}{4} \right\rangle$$

$df(a) : \mathbb{R}^2 \to \mathbb{R}$

lineár.

$$df[h_1, h_2] = \operatorname{grad} f|_a \cdot \langle h_1, h_2 \rangle = \left(\tfrac{1}{2}, -\tfrac{1}{4}\right)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \underline{\underline{\tfrac{1}{2} h_1 - \tfrac{1}{4} h_2}}$$

$h_1 = (x - x_0)$

$h_2 = (y - y_0)$

$\vec{a} = (x_0, y_0)$

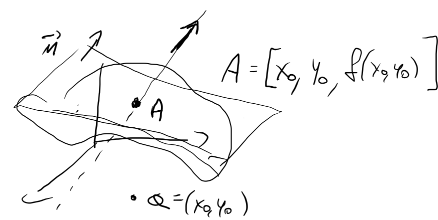
tečná rovina $\quad z = f(a) + \operatorname{grad} f|_a (x - x_0, y - y_0)$

$$z = 1 + \tfrac{1}{2}(x-2) - \tfrac{1}{4}(y-1) \qquad \vec{n} = \left\langle \tfrac{1}{2}, -\tfrac{1}{4}, -1 \right\rangle$$

$A = [2,1,1] \qquad \vec{n} = \langle 1/2, -1/4, -1 \rangle$

$\to p : \quad A + t\vec{n} \quad t \in \mathbb{R}$

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ -1/4 \\ -1 \end{pmatrix}}_{} \quad t \in \mathbb{R}$$



$ax + by + cz + d = 0$

$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

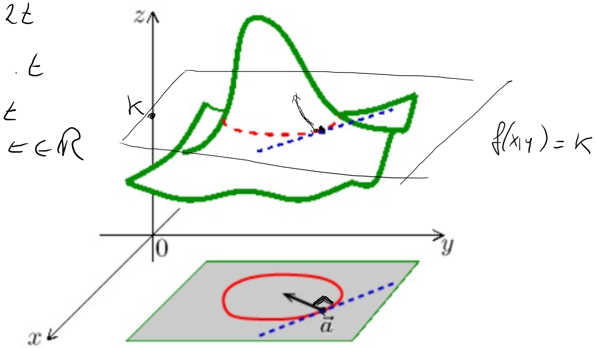
Napište rovnici tečné roviny ke grafu funkce $f(x, y) = xy$, která je kolmá na přímku p:

$$\frac{x+2}{2} = \frac{y+2}{1} = \frac{z-1}{-1}. \iff \begin{cases} \frac{x+2}{2} = t \\ \frac{y+2}{1} = t \\ \frac{z-1}{-1} = t \end{cases} \quad t \in \mathbb{R} \qquad \begin{cases} x = -2 + 2t \\ y = -2 + t \\ z = 1 - t \end{cases} \quad t \in \mathbb{R}$$

$$\Updownarrow$$

$$\begin{cases} \frac{x+2}{2} = \frac{y+2}{1} \\ \frac{x+2}{2} = \frac{z-1}{-1} \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}}_{A} + t \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}_{\vec{v}}$$



Má-li funkce f na okolí bodu $\vec{a}$ spojité první parciální derivace,

pak je gradient $\nabla f(\vec{a})$ kolmý na hladinu konstantnosti procházející bodem $\vec{a}$.

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$\vec{m} = \lambda \cdot \vec{v} = \lambda \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\bar{f}(x, y, z) = xy - z$$

$$\text{Graf } f = \{(x,y,z) : \underline{z = xy}\} = \underset{(K=0)}{\text{vrstevnice}} \ \bar{f} = \{(x,y,z) : \underset{xy - z = 0}{\bar{f} = 0}\}$$

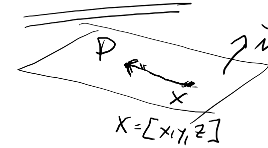
$$\text{grad } \bar{f} = \langle y, x, -1 \rangle$$

$$(x_0, y_0, z_0) : \quad \underline{\text{grad } \bar{f}|_{(x_0,y_0,z_0)} = \lambda \cdot \langle 2, 1, -1 \rangle}$$

$$\begin{cases} y = 2\lambda \\ x = \lambda \\ -1 = -\lambda \end{cases} \quad \left.\begin{matrix} y = 2 \\ x = 1 \end{matrix}\right\} \Rightarrow z = 2$$

$$\lambda = 1$$

$$\underline{\underline{P\,[1, 2, 2]}}$$



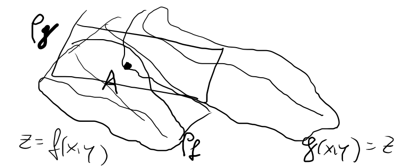
tečná rovina : $\vec{m} \cdot (X - P) = 0$

$$\vec{m} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad (2, 1, -1)(x-1, y-2, z-2) = 0$$

$$\underline{\underline{2(x-1) + (y-2) - (z-2) = 0}}$$

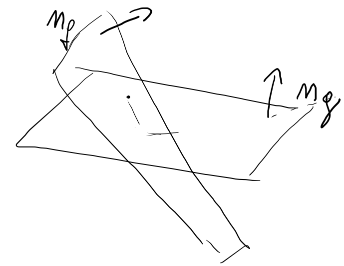
Nalezněte úhel, který v bodě $A(1,0,0)$ svírají grafy funkcí $f(x,y)=\ln(\sqrt{x^2+y^2})$ a $g(x,y)=\sin(xy)$.

$z=\ln\sqrt{x^2+y^2}$ $\qquad$ $z=\sin(xy)$



$\bar{f}(x,y,z)=\ln\sqrt{x^2+y^2}-z \qquad \text{Graf } f=\{(x,y,z):\ \bar{f}=0\}$

$\bar{g}(x,y,z)=\sin(xy)-z \qquad \text{Graf } g=\{(x,y,z):\ \bar{g}=0\}$



$\nabla\bar{f}=\text{grad }\bar{f}=\left\langle \frac{1}{2}\frac{2x}{x^2+y^2},\ \frac{y}{x^2+y^2},\ -1\right\rangle$

$\nabla\bar{f}\big|_{(1,0,0)}=\langle 1,0,-1\rangle=\vec{n}_f$

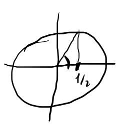
$\ln\sqrt{x^2+y^2}=\ln(x^2+y^2)^{1/2}$

$=\frac{1}{2}\ln(x^2+y^2)$

$\nabla\bar{g}=\langle y\cos xy,\ x\cos xy,\ -1\rangle$

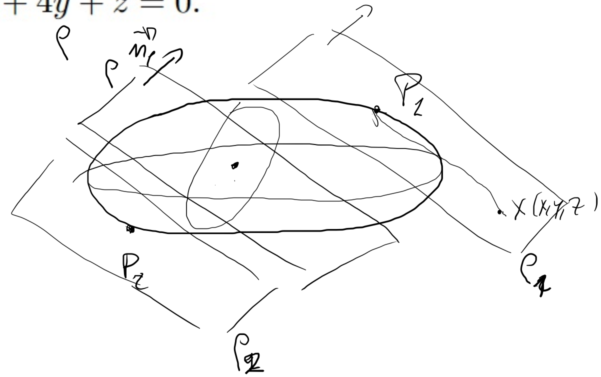
$\nabla\bar{g}\big|_{(1,0,0)}=\langle 0,1,-1\rangle=\vec{n}_g$

$\alpha\in\langle 0,\pi/2\rangle:\qquad \cos\alpha=\frac{|\vec{n}_f\cdot\vec{n}_g|}{\|\vec{n}_f\|\cdot\|\vec{n}_g\|}=\frac{0+0+1}{\sqrt{2}\cdot\sqrt{2}}=\frac{1}{2}$



$\alpha=\pi/3.$

Nalezněte rovnici tečné roviny k elipsoidu $x^2 + 2y^2 + z^2 = 13$, která je rovnoběžná s rovinou $2x + 4y + z = 0$.



$$\text{elip.} = \{(x,y,z) : f(x,y,z) = 0\}_{k=0}$$

$$f(x,y,z) = x^2 + 2y^2 + z^2 - 13$$

$$\nabla f|_P \perp \text{elips.} \qquad P \in \text{elip.}$$

$$\nabla f = \langle 2x, 4y, 2z \rangle \qquad \vec{n}_\rho = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \qquad \nabla f \parallel \vec{n}_\rho$$

$$\nabla f = \lambda \cdot \vec{n}_\rho$$

$$\begin{cases} 2x = 2\lambda & x = \lambda \\ 4y = 4\lambda & y = \lambda \\ 2z = \lambda & z = \lambda/2 \\ x^2 + 2y^2 + z^2 = 13 & \lambda^2 + 2\lambda^2 + \frac{\lambda^2}{4} = 13 \quad \frac{13}{4}\lambda^2 = 13 \quad \lambda^2 = 4 \quad \lambda = \pm 2 \end{cases}$$

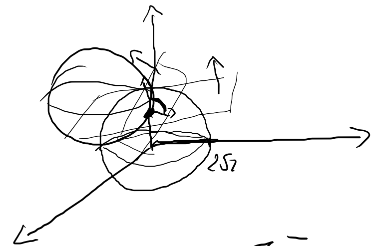
$P_1\,[2, 2, 1]$ $\quad P_2\,[-2, -2, -1]$

$\rho_1 \quad 2x + 4y + z = 13$

$\rho_2 \quad \vec{n}_\rho \cdot (X - P_2) = 0 \qquad 2(x+2) + 4(y+2) + (z+1) = 0$

Najděte úhel sevřený dvěma plochami $x^2 + y^2 + z^2 = 8$ a $(x-1)^2 + (y-2)^2 + (z-3)^2 = 6$ v bodě $a = (2, 0, 2)$.

(x,y,z) $(0,0,0)$



$\nabla \bar{f}|_a$

$\nabla \bar{g}|_a$

$\bar{f}(x,y,z) = x^2 + y^2 + z^2 - 8$

$\bar{g}(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2 - 6$

$\nabla \bar{f}|_a = \langle 2x, 2y, 2z \rangle|_{(2,0,2)} = \langle 4, 0, 4 \rangle$

$\nabla \bar{g}|_a = \langle 2(x-1), 2(y-2), 2(z-3) \rangle|_{(2,0,2)} = \langle 2, -4, -2 \rangle$

$\cos\alpha = \frac{|8+0-8|}{\|\cdot\| \cdot \|\cdot\|} = 0$ $\qquad \alpha = \pi/2$

Najděte linearizaci funkce $f(x,y,z) = e^{xy^2} + x^4yz$ v bodě $(1,1,1) = a_0 = (x_0, y_0, z_0)$

$$\ell(a_0 + \vec{h}) = f(a_0) + \underbrace{df(a_0)[\vec{h}]}_{diff.} \qquad \vec{h} = (h_1, h_2, h_3)$$

$$\nabla f\big|_{a_0} = \langle y^2 e^{xy^2} + 4x^3yz,\ 2xy\, e^{xy^2} + x^4 z,\ x^4 y \rangle\big|_{(1,1,1)} = \langle e+4,\ 2e+1,\ 1 \rangle$$

$$f(1,1,1) = e + 1$$

$$\ell(a_0 + \vec{h}) = e + 1 + (e+4)h_1 + (2e+1)h_2 + h_3$$

$h_1 = x - x_0$

$h_2 = y - y_0$

$h_3 = z - z_0$

$$\ell(x,y,z) = e + 1 + (e+4)(x-1) + (2e+1)(y-1) + (z-1)$$

Najděte Taylorův polynom druhého stupně pro funkci f v okolí bodu a.

(a) $f(x,y) = \frac{1}{x-y}$, $a = (2,1)$,

(b) $f(x,y) = x^2y^3 - 2x^4 + y^2$, $a = (0,0)$,

(c) $f(x,y,z) = xy^2z^3$, $a = (1,2,1)$.

$$T_2(a_0 + \vec{h}) = \overbrace{f(a_0) + df(a_0)[\vec{h}]} + \frac{1}{2!}d^2f(a_0)[\vec{h},\vec{h}]$$

kde $\vec{h} \in \mathbb{R}^n$ a bilinearni forma $d^2f(a_0) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ je urcena tzv. Hessovou matici

$$\mathbb{A} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(a_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a_0) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a_0) \end{pmatrix}$$

jako

$$d^2f(a_0)[\vec{u},\vec{v}] = \vec{u}^T \cdot \mathbb{A} \cdot \vec{v} \quad \text{pro} \quad \vec{u},\vec{v} \in \mathbb{R}^n$$

Dú 3

Nalezněte tečnou rovinu k ploše $S: \ x^2 + 2y^2 + 3z^2 = 6$, která je kolmá na roviny $2x - y + z = 0$ a $2x - y - 5z = 0$.